

SOLUTIONS TO COOPERATIVE GAMES WITHOUT SIDE PAYMENTS

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An extension of Von Neumann Morgenstern solution theory to cooperative games without side payments has been outlined in [1]. In this paper we revise some of the definitions given in [1] and prove that in the new theory every three-person constant sum game is solvable (see [1, Theorem 1]). Other results that were formulated in [1] had already been proved in [2]. [1; 2] are also necessary for a full understanding of the basic definitions of this paper.

1. Basic definitions. If N is a set with n members, we denote by E^N the n -dimensional euclidean space the coordinates of whose points are indexed by the members of N . Subsets of N will be denoted by S . If $x \in E^N$ and $i \in N$, x^i will denote the coordinate of x corresponding to i ; x^S will denote the set $\{x^i: i \in S\}$. The superscript N will be omitted, thus we write x instead of x^N . We write $x^S \geq y^S$ if $x^i \geq y^i$ for all $i \in S$; similarly for $>$ and $=$. $\emptyset$ denotes the empty set.

DEFINITION 1.1. An n -person characteristic function is a pair (N, v) where N is a set with n members, and v is a function that carries each $S \subset N$ into a set $v(S) \subset E^N$ so that

- (1) $v(S)$ is closed,
- (2) $v(S)$ is convex,
- (3) $v(\emptyset) = E^N$,
- (4) if $x \in v(S)$ and $x^S \geq y^S$ then $y \in v(S)$.

DEFINITION 1.2. An n -person game is a triad (N, v, H) , where (N, v) is an n -person characteristic function and H is a convex compact subset of $v(N)$.

We notice that this definition is not identical with that given in [1; 2]. In the first place v is not assumed to be superadditive, i.e., the condition: $v(S_1 \cup S_2) \supset v(S_1) \cap v(S_2)$ for every pair of disjoint coalitions S_1 and S_2 is dropped. Secondly H need not be a polyhedron.

2. Solutions. Let $G = (N, v, H)$ be an n -person game.

DEFINITION 2.1. Let $x, y \in E^N$, $S \neq \emptyset$. x dominates y via S , written $x \succ_S y$, if $x \in v(S)$ and $x^S > y^S$.

DEFINITION 2.2. x dominates y , written $x \succ y$, if there is an S such that $x \succ_S y$.

For $x \in E^N$ the following sets are defined: $\text{dom}_S x = \{y: x \succ_S y\}$ and $\text{dom } x = \{y: x \succ y\}$. Let $K \subset E^N$. We define $\text{dom}_S K = \bigcup_{x \in K} \text{dom}_S x$ and $\text{dom } K = \bigcup_{x \in K} \text{dom } x$.

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DEFINITION 2.3. V is K -stable if $V = K - \text{dom } V$.

DEFINITION 2.4. The K -core is the set $K - \text{dom } K$.

We use the following abbreviation: P.S.O.—the proof, which is straightforward, will be omitted.

PROPOSITION 2.5. Every K -stable set contains the K -core. P.S.O.

PROPOSITION 2.6. If for each $x \in K \cap \text{dom } K$ there is a $y \in K - \text{dom } K$ such that $y \succ x$ then the K -core is the only K -stable set. P.S.O.

We denote: $v^i = \sup_{x \in v(\{i\})} x^i$.

DEFINITION 2.7. x is individually rational if $x^i \geq v^i$ for all $i \in N$.

DEFINITION 2.8. x is group rational if there is no $y \in H$ such that $y > x$.

We denote: $\bar{A} = \{x: x \in H, x \text{ is individually rational}\}$ and $A = \{x: x \in \bar{A}, x \text{ is group rational}\}$.

PROPOSITION 2.9. K is $\bar{A}$ -stable if and only if it is A -stable.

Proof. Let K be $\bar{A}$ -stable. We show firstly that (1) $\bar{A} - A \subset \text{dom } K$. If $x \in \bar{A} - A$ then there is a $y_0 \in \bar{A}$ such that $y_0 > x$. Define $f(y) = \min_{i \in N} (y^i - x^i)$. Since f is continuous and $\bar{A}$ is compact f receives its maximum in $\bar{A}$ at a point z , which must be in A . By 1.2 $z \in v(N)$. $f(z) \geq f(y_0)$; therefore $z > x$. We have that $z \succ_N x$ and if $w \succ z$ then $w \succ x$. If $z \in K$ then $x \in \text{dom } K$. If $z \in \text{dom } K$ then there is a $w_0 \in K$ such that $w_0 \succ z$ and therefore $w_0 \succ x$, so $x \in \text{dom } K$. From (1) it follows that $\bar{A} - \text{dom } K = A - \text{dom } K$ and therefore K is A -stable. Now, let K be A -stable. If $x \in \bar{A} - A$ we define z as before and we see that $z \in A \subset K \cup \text{dom } K$ implies that $x \in \text{dom } K$. We conclude that (1) holds and therefore $K = A - \text{dom } K = \bar{A} - \text{dom } K$, i.e., K is $\bar{A}$ stable.

DEFINITION 2.10. A solution of G is an A -stable set.

If G has a solution we say that G is solvable.

THEOREM 2.11. Every two person game has a unique solution, consisting of all of A . P.S.O.

DEFINITION 2.12. G is constant-sum if H is contained in a plane

$$\sum_{i \in N} x^i = e.$$

3. Three-person constant sum games.

I. Auxiliary lemmas. We use the following abbreviations: 3-P.C.G. — three-person constant sum game, W. L. G. — without loss of generality.

Let $G = (N, v, H)$ be 3-P.C.G. We denote the members of N by the first three positive integers and set $S_i = N - \{i\}$ for $i = 1, 2, 3$. Let $x \in H$. We denote: $\sum_{i=1}^3 x^i = e$ and $L = \{y: \sum_{i=1}^3 y^i = e\}$. We have that $\bar{A} = \{x: x \in H, x^i \geq v^i, i = 1, 2, 3\} = A$. So A is a convex compact subset of L . Domination between

points of A is possible only via the S_i , i.e., if $x, y \in A$ and $x \prec_{S_i} y$ then S is one of the S_i . For a subset B of L and $i \in N$ the following sets are defined: $B^i = B \cap v(S_i)$, $\hat{B}^i = B^i \cap \text{dom}_{S_i} B^i$ and $b^i = B^i - \hat{B}^i$.

LEMMA I.1. *If B is convex then $\hat{B}^i$ is convex.*

Proof. $B^i = B \cap v(S_i)$ is convex. If $x_1, x_2 \in \hat{B}^i$ then there are $y_1, y_2 \in B^i$ such that $y_j \prec_{S_i} x_j$ for $j = 1, 2$. If $0 < t < 1$, $x = tx_1 + (1-t)x_2$ and $y = ty_1 + (1-t)y_2$ then $x, y \in B^i$ and $y \prec_{S_i} x$, so $x \in \hat{B}^i$.

We remark that A^i is convex and compact, $\hat{A}^i$ is convex and a^i is compact.

Let $x \in L$ and $\varepsilon > 0$. The set $\{y: y \in L, \sum_{i=1}^3 (y^i - x^i)^2 < \varepsilon^2\}$ is denoted by $S(x, \varepsilon)$. x is an interior point of a subset B of L if there is an $\varepsilon > 0$ such that $S(x, \varepsilon) \subset B$.

LEMMA I.2. *If $B \subset L$ is convex and $K = \hat{B}^i \cap \hat{B}^j \neq \emptyset$, $i \neq j$, then K contains an interior point.*

Proof. W.L.G. $i = 1$ and $j = 2$. We show firstly that $K \neq \emptyset$ implies that B contains an interior point. If B has no interior points then there are points x_1 and x_2 such that every $y \in B$ can be written as $y = tx_1 + (1-t)x_2$, $-\infty < t < \infty$. Let $x \in K$. $x = t_0x_1 + (1-t_0)x_2$. There are $y_l = t_lx_1 + (1-t_l)x_2$, $y_l \prec_{S_l} x$ for $l = 1, 2$. We have $y_1^2 > x^2$ and $y_1^3 > x^3$, i.e., $t_1x_1^2 + (1-t_1)x_2^2 > t_0x_1^2 + (1-t_0)x_2^2$ and $t_1x_1^3 + (1-t_1)x_2^3 > t_0x_1^3 + (1-t_0)x_2^3$. So $(t_1 - t_0)(x_1^2 - x_2^2) > 0$ and $(t_1 - t_0)(x_1^3 - x_2^3) > 0$. Therefore $\text{sgn}(x_1^2 - x_2^2) = \text{sgn}(x_1^3 - x_2^3)$. In the same way $y_2 \prec_{S_2} x$ implies that $\text{sgn}(x_1^2 - x_2^2) = \text{sgn}(x_1^3 - x_2^3)$. So the three differences $x_1^k - x_2^k$ have the same sign, which is impossible since $\sum_{k=1}^3 x_1^k = \sum_{k=1}^3 x_2^k$. Now, let z be an interior point of B and $y \in K$. For small positive t the points $tz + (1-t)y$ are interior points of K .

LEMMA I.3. *If $x \in O = \bigcap_{i=1}^3 \hat{A}^i$ then x is an interior point of O .*

Proof. There are $y_j \prec_{S_j} x$ for $j = 1, 2, 3$. We have: $y_1^2 > x^2$, $y_1^3 > x^3$, $y_2^1 > x^1$, $y_2^3 > x^3$, $y_3^1 > x^1$ and $y_3^2 > x^2$. There exist $0 < t_k < 1$ such that $z_k = t_k y_1 + (1-t_k) y_k$ satisfy $z_k^1 = x^1$, $k = 2, 3$. Since $z_2^3 > x^3$ and $z_3^3 < x^3$ there is a $0 < t_1 < 1$ such that $x = t_1 z_2 + (1-t_1) z_3$. So x is an interior point of the convex hull of $\{y_1, y_2, y_3\}$ and therefore of A . But if $x \in O$ is an interior point of A then x is also an interior point of O .

LEMMA I.4. *If $B \subset L$ is convex, $x_1, x_2 \in B$, $x_1^k = x_2^k$, $x_1^i < x_2^i$, $x_1^j > x_2^j$ and y satisfies $y^i = x_2^i$, $y^j = x_1^j$ and $y^k = x_1^i + x_1^k - x_2^i$ then: $y \notin B$ if and only if $B \cap \{z: z^{S_k} \geq y^{S_k}\} = \emptyset$. P.S.O.*

LEMMA I.5. *If $x \in \hat{A}^i$ then there is a $y \in a^i$ such that $y \prec_{S_i} x$ and for every $\varepsilon > 0$, $S(y, \varepsilon) \cap \hat{A}^i \neq \emptyset$.*

Proof. Define $f(z) = \min_{j \in S_i} (z^j - x^j)$. f receives its maximum in A^i at a point

y which must be in A^i . Since $x \in \mathring{A}^i$ $f(y) > 0$ and therefore $y \succ_{S_1} x$. If $0 \leq t < 1$ then $ty + (1-t)x$ is in $\mathring{A}^i$, therefore for every $\varepsilon > 0$ $S(y, \varepsilon) \cap \mathring{A}^i \neq \emptyset$.

If $x, y \in L$ then the set $\{z: z = tx + (1-t)y, 0 \leq t \leq 1\}$ is denoted by $[xy]$ and is called an *interval*. x and y are called the *ends* of $[xy]$. $(xy) = [xy] - \{x\}$. $[xy] = [xy] - \{y\}$. $(xy) = [xy] - (\{x\} \cup \{y\})$. For $i = 1, 2, 3$ the following sets are defined: $D^i = A^j \cap A^k$, where $S_i = \{j, k\}$, and $F^i = \{x: x \in D^i, x^i \geq y^i \text{ for every } y \in D^i\}$. D^i is convex and compact. F^i is an interval.

LEMMA I.6. Let $S_k = \{i, j\}$. If x^i receives its maximum in F^k at a point α , then: $\alpha \notin a^j$ if and only if $\mathring{A}^j \supset A^i$.

Proof. W.L.G. $i = 2$ and $j = 3$. If $\mathring{A}^3 \supset A^2$ then $\alpha \in F^1 \subset D^1 \subset A^2 \subset \mathring{A}^3$. If $\alpha \notin a^3$ then there is an $x \in A^3$ such that $x \succ_{S_3} \alpha$. There is an $\varepsilon > 0$ such that $U = S(\alpha, \varepsilon) \cap A \subset \mathring{A}^3$. Now we show that if $y \in A^2$ then $y^1 \leq \alpha^1$. If there is a $z \in A^2$ such that $z^1 > \alpha^1$ then for a small positive $tu = tz + (1-t)\alpha$ satisfies $u^1 > \alpha^1$ and $u \in U$. So we have $u \in D^1$ and $u^1 > \alpha^1$ which is impossible. Next we show that $y \in A^2$ implies that $y^2 \leq \alpha^2$. Suppose that there is a $z \in A^2$ such that $z^2 > \alpha^2$. If $z^1 = \alpha^1$ then for a small positive $tu = tz + (1-t)\alpha$ satisfies $u \in U$, $u^1 = \alpha^1$ and $u^2 > \alpha^2$. So we have that $u \in F^1$ and $u^2 > \alpha^2$ which is impossible. If $z^1 < \alpha^1$ then there is a $0 < t < 1$ such that $w = tz + (1-t)x$ satisfies $w^1 = \alpha^1$ and $w^2 > \alpha^2$. $\alpha^{S_2} \geq w^{S_2}$ therefore $w \in A^2$, but this is impossible as we have already shown. We have shown that every $y \in A^2$ satisfies $y^{S_3} \leq \alpha^{S_3}$. Since $\alpha \in \text{dom}_{S_3} x$ we have $A^2 \subset A \cap \text{dom}_{S_3} x \subset \mathring{A}^3$.

The sets $\{A^1, A^2\}$, $\{A^2, A^3\}$ and $\{A^3, A^1\}$ will be called *pairs*.

DEFINITION I.7. The pair $\{A^i, A^j\}$ intersects maximally if:

(1) $\mathring{A}^i \cap \mathring{A}^j \neq \emptyset$.

(2) $a^i \cap a^j \neq \emptyset$.

The number of pairs that intersect maximally will be denoted by $m(G)$.

LEMMA I.8. Let $i \neq j$ and $\mathring{A}^i \cap \mathring{A}^j \neq \emptyset$. $\mathring{A}^i \not\supset A^j$ and $\mathring{A}^j \not\supset A^i$ if and only if $\{A^i, A^j\}$ intersects maximally.

Proof. W.L.G. $i = 2$ and $j = 3$. If $\mathring{A}^3 \supset A^2$ or $\mathring{A}^2 \supset A^3$ then $a^2 \cap a^3 = \emptyset$ and therefore $\{A^2, A^3\}$ does not intersect maximally. Now suppose that $\mathring{A}^2 \not\supset A^3$ and $\mathring{A}^3 \not\supset A^2$. Let x^2 and x^3 receive their maxima in F^1 at the points α and β respectively. By I.6: $A^2 \not\supset \mathring{A}^3$ implies that $\alpha \in a^3$ and $\mathring{A}^2 \not\supset A^3$ implies that $\beta \in a^2$. We have (1) that $F^1 \subset a^2 \cup a^3$ and $F^1 \cap a^k \neq \emptyset$, $k = 2, 3$. Since F^1 is connected and $F^1 \cap a^3$ and $F^1 \cap a^2$ are closed we must have $(F^1 \cap a^2) \cap (F^1 \cap a^3) = F^1 \cap a^2 \cap a^3 \neq \emptyset$.

From the proof of I.8 we can conclude that: (I.9) if $\{A^i, A^j\}$ intersects maximally then $F^k \cap a^i \cap a^j \neq \emptyset$ where $\{k\} = N - \{i, j\}$.

(1) Otherwise there is $x \in F^1 \cap \mathring{A}^2 \cap \mathring{A}^3$. Let $z \succ_{S_2} x$. For small $t > 0$, $u = tz + (1-t)x$ satisfy $u^1 > x^1$ and $u \in D^1$, which is impossible.

LEMMA I.10. *If $i \neq j$, $x, y \in a^i$, $x \neq y$ and $x^j = y^j$ then every $z \in A^i$ satisfies $z^j \leq x^j$ and $\{u: u \in A^i, u^j = x^j\} \subset a^i$. P.S.O.*

LEMMA I.11. *Let $S_k = \{i, j\}$. If $\{A^i, A^j\}$ intersects maximally and x^i and x^j take their maxima in F^k at the points α and β respectively then $F^k = [\alpha\beta]$ and one of the following possibilities holds:*

- | | | |
|---|---|-----------------------------|
| (a) $\alpha = \beta$, | $\alpha \in a^i \cap a^j$ | |
| (b) $\alpha \neq \beta$, | $[\alpha\beta] \subset a^i \cap a^j$, | |
| (c _j) $\alpha \neq \beta$, | $(\alpha\beta) \subset a^i \cap \overset{\circ}{A}^j$, | $\alpha \in a^i \cap a^j$, |
| (c _i) $\alpha \neq \beta$, | $[\alpha\beta] \subset a^j \cap \overset{\circ}{A}^i$, | $\beta \in a^i \cap a^j$. |

Proof. W.L.G. $i = 2$ and $j = 3$. We saw in the proof of I.8 that $\alpha \in F^1 \cap a^3$ and $\beta \in F^1 \cap a^2$. If $\alpha = \beta$ then (a) holds. If $\alpha \neq \beta$ we have the following possibilities for the relative positions of a^3 and F^1 :

- (1) There is no $x \neq \alpha$ in $a^3 \cap F^1$, i.e., $(\alpha\beta) \subset \overset{\circ}{A}^3$.
 (2) There is an $x \neq \alpha$ in $a^3 \cap F^1$, and, therefore, by I.10, $F^1 \subset a^3$. And similarly for a^2 and F^1 :

- (3) There is no $y \neq \beta$ in $a^2 \cap F^1$, i.e., $[\alpha\beta] \subset \overset{\circ}{A}^2$.
 (4) There is a $y \neq \beta$ in $a^2 \cap F^1$ and therefore $F^1 \subset a^2$.

Since $F^1 \subset a^2 \cup a^3$ (1) and (3) cannot hold together. If (2) and (4) hold together then we have (b). If (1) and (4) hold together, then we have (c_j). If (2) and (3) hold together then we have (c_i). We say that F^k has a -shape if (a) holds; similarly for (b), (c_i) and (c_j).

For $x \in A$ the following sets are defined: $Q_i(x) = \{y: y \in A, y^{S_i} \geq x^{S_i}\}$, $T_i(x) = \{y: y \in A, y^i < x^i\}$ and $R_i(x) = A - T_i(x)$. We remark that:

- (I.12) $\text{dom } Q_i(x) \cap R_i(x) = \text{dom}_{S_i} Q_i(x) \cap R_i(x)$,
 (I.13) $x \notin \overset{\circ}{A}^i$ if and only if $x \in Q_i(x) - \text{dom } Q_i(x)$,
 (I.14) $x \notin \overset{\circ}{A}^i$ if and only if $Q_i(x) \cap \overset{\circ}{A}^i = \emptyset$.

LEMMA I.15. *If $x \in A - \overset{\circ}{A}^i$, $y \neq x$, $y \in A^i \cap Q_i(x)$ then there is a $j \in S_i$ such that every $z \in A^i$ satisfies $z^j \leq x^j$. P.S.O.*

LEMMA I.16. *Let $S_k = \{i, j\}$. We denote the ends of F^k by α and β such that $\alpha^i \geq \beta^i$. If $\gamma \in A$ satisfies $\gamma^k = \alpha^k$ and $\gamma^i < \beta^i$ then $Q_i(\gamma) \cap A^i = \emptyset$. P.S.O.*

LEMMA I.17. *Let $S_k = \{i, j\}$. If $\gamma \in F^k \cap a^i \cap a^j$ and $x \in R_k(\gamma) - F^k$ then $\text{dom } x \cap Q_k(\gamma) = \emptyset$.*

Proof. W.L.G. $i = 2$ and $j = 3$. We denote the ends of F^1 by α and β such that $\alpha^2 \geq \beta^2$. Let $x \in R_1(\gamma) - F^1$ and $y \in Q_1(\gamma)$. $x^2 + x^3 \leq \gamma^2 + \gamma^3 \leq y^2 + y^3$ so $x \succ_{S_1} y$ is impossible. If $x^1 > \gamma^1$ and $x \succ_{S_2} y$ or $x \succ_{S_3} y$ then $x \succ_{S_2} \gamma$ or $x \succ_{S_3} \gamma$ respectively, which is impossible. If $x^1 = \gamma^1$ then either $x^2 < \beta^2$ or $x^3 < \alpha^3$. If $x^3 < \alpha^3$ then, by I.16, $x \notin A^3$. Since $y^3 \geq \gamma^3 \geq \alpha^3 > x^3$, if $x \succ y$ then $x \succ_{S_3} y$, but this is impossible. Similarly if $x^2 < \beta^2$ then $x \succ y$ is impossible.

LEMMA I.18. If a $B \subset A$ is convex and compact then (N, v, B) is 3-P.C.G., $B^i = A^i \cap B$ and $\dot{B}^i \subset \dot{A}^i \cap B$. P.S.O.

If $B \subset A$ is convex and compact we say that B is solvable or that B has a solution if (N, v, B) is solvable. We also write $m(B)$ instead of $m((N, v, B))$.

LEMMA I.19. If $x \in D^k$ and $l \in S_k$ then $\dot{R}_k^l(x) = \dot{A}^l \cap R_k(x)$.

Proof. If $y \in \dot{A}^l \cap R_k(x)$ then $y^k \geq x^k$ and there is a $z \in A^l$ such that $z \succ_{S_1} y$. Since $z^k > y^k$ $z \in R_k(x)$. So we have $z \in R_k(x) \cap A^l = R_k^l(x)$ and therefore $y \in \dot{R}_k^l(x)$. We have shown that $\dot{R}_k^l(x) \supset \dot{A}^l \cap R_k(x)$. By I.18 $\dot{R}_k^l(x) \subset \dot{A}^l \cap R_k(x)$, so $\dot{R}_k^l(x) = \dot{A}^l \cap R_k(x)$.

LEMMA I.20. If $x \in D^k - \dot{A}^k$, $l \in S_k$ and $\{A^k, A^l\}$ does not intersect maximally, then $\{R_k^l(x), R_k^k(x)\}$ does not intersect maximally.

Proof. Since $\{A^k, A^l\}$ does not intersect maximally, by I.8 at least one of the following possibilities holds: $\dot{A}^k \cap \dot{A}^l = \emptyset$, $\dot{A}^l \supset \dot{A}^k$ or $\dot{A}^k \supset \dot{A}^l$. If $\dot{A}^k \cap \dot{A}^l = \emptyset$ then $\dot{R}_k^l(x) \cap \dot{R}_k^k(x) \subset \dot{A}^k \cap \dot{A}^l \cap R_k(x) = \emptyset$. If $\dot{A}^l \supset \dot{A}^k$ then $\dot{R}_k^l(x) = \dot{A}^l \cap R_k(x) \supset \dot{A}^k \cap R_k(x) = R_k^k(x)$. $\dot{A}^k \supset \dot{A}^l$ is impossible since $x \in A^l - \dot{A}^k$.

DEFINITION I.21. Let $B_1, \dots, B_l$ be convex compact subsets of A . $B_1, \dots, B_l$ are called *independent* if there exist solutions $V_1, \dots, V_l$, V_i solution of B_i respectively, such that $\text{dom } V_k \cap (\bigcup_{j=1}^l V_j) = \emptyset$ for $k = 1, \dots, l$.

LEMMA I.22. If $B_1, \dots, B_l$ are independent then there exist solutions $V_1, \dots, V_l$, V_i solution of B_i for $i = 1, \dots, l$, such that $\bigcup_{j=1}^l V_j$ is $\bigcup_{j=1}^l B_j$ -stable. P.S.O.

In the following three subsections we shall prove:

THEOREM. Every 3-P.C.G. G is solvable.

The proof will be by induction on $m(G)$.

II. *First part:* $m(G) = 0$. In this subsection we show that every 3-P.C.G. G for which $m(G) = 0$ is solvable. We also prove some additional auxiliary lemmas:

LEMMA II.1. Let G be 3-P.C.G. If $\dot{A}^1 \cap \dot{A}^2 = \dot{A}^2 \cap \dot{A}^3 = \dot{A}^3 \cap \dot{A}^1 = \emptyset$ then the A -core is the solution of G .

Proof. Denote $C = A - \text{dom } A$. If $x \in A - C$ then there is a $y \in A$ that dominates it. There is an i such that $y \succ_{S_i} x$, i.e., $x \in \dot{A}^i$. By I.5 there is a $z \in A^i$ such that $z \succ_{S_1} x$. If $z \notin C$ then $z \in \dot{A}^l$ where $l \neq i$. There is an $\varepsilon > 0$ such that $S(z, \varepsilon) \cap A \subset \dot{A}^l$. But $S(z, \varepsilon) \cap \dot{A}^i \neq \emptyset$; therefore $\dot{A}^l \cap \dot{A}^i \neq \emptyset$ which is impossible. We have shown that for every $x \in A - C$ there is a $z \in C$ such that $z \succ x$. By 2.6 C is the only A -stable set.

LEMMA II.2. Let G be 3-P.C.G. If $m(G) = 0$ then the A -core is the solution of G .

Proof. If $\dot{A}^1 \cap \dot{A}^2 = \dot{A}^2 \cap \dot{A}^3 = \dot{A}^3 \cap \dot{A}^1 = \emptyset$ then by II.1 the A -core is the solution of G . If it is not the case then, W.L.G., we assume that $\dot{A}^2 \cap \dot{A}^3 \neq \emptyset$. Since $\{A^2, A^3\}$ does not intersect maximally we have that either $\dot{A}^3 \supset \dot{A}^2$ or $\dot{A}^2 \supset \dot{A}^3$.

W.L.G. we suppose that $\dot{A}^3 \supset A^2$. There are three possibilities for the relative position of A^1 and A^3 : (a) $\dot{A}^1 \cap \dot{A}^3 = \emptyset$, (b) $\dot{A}^3 \supset A^1$ or (c) $\dot{A}^1 \supset A^3$. In each case we show that C , the A -core, is the solution of G . (a) $\dot{A}^1 \cap \dot{A}^3 = \emptyset$. If $x \in A - C$ then there is a $y \in A$ such that $y \prec_{S_1} x$ for some i . Since $\dot{A}^3 \supset A^2$ we may assume that $i \in S_2$. There is a $z \in a^i$ such that $z \prec_{S_1} x$. If $i = 1$ then, since $\dot{A}^1 \cap \dot{A}^j = \emptyset$ for $j = 2, 3$, $z \in C$. If $i = 3$ then, since $a^3 \cap \dot{A}^2 = \emptyset$ and $\dot{A}^1 \cap \dot{A}^3 = \emptyset$, $z \in C$. By 2.6 C is the solution of G . (b) $\dot{A}^3 \supset A^1$. If $x \in A - C$ then there is a $y \in A$ such that $y \prec_{S_3} x$. So there is a $z \in a^3$ such that $z \prec x$. Since $a^3 \cap (\dot{A}^1 \cup \dot{A}^2) = \emptyset$, $z \in C$. So C is the solution of G . (c) $\dot{A}^1 \supset A^3$. The proof in this case parallels that in case (b).

Let $G = (N, v, H)$ be 3-P.C.G.

LEMMA II.3. *If $\xi \in D^k - \dot{A}^k$, $\eta \in D^k \cap Q_k(\xi)$ and U is a solution of $Q_k(\eta)$ then $V = U \cup [\xi\eta]$ is a solution of $Q_k(\xi)$.*

Proof. W.L.G. $k = 1$. $\eta^1 \leq \xi^1$, $\eta^2 \geq \xi^2$ and $\eta^3 \geq \xi^3$; therefore $(\text{dom}_{S_3} [\xi\eta] \cup \text{dom}_{S_2} [\xi\eta]) \cap [\xi\eta] = \emptyset$. $\xi \cdots \dot{A}^1$; therefore $Q_1(\xi) \cap \dot{A}^1 = \emptyset$. So $\text{dom}_{S_1} [\xi\eta] \cap [\xi\eta] = \emptyset$. Summing we have (1) $\text{dom} [\xi\eta] \cap [\xi\eta] = \emptyset$. Now we show (2) $Q_1(\xi) - [\xi\eta] - Q_1(\eta) \subset \text{dom} [\xi\eta]$. Let $x \in Q_1(\xi) - [\xi\eta] - Q_1(\eta)$. If $x^1 \geq \eta^1$ then there is a $y \in [\xi\eta]$ such that $y^1 = x^1$. $y \neq x$ so we may assume that $y^2 > x^2$. Under this assumption we can find a $z \in [\xi\eta]$ with $z^{S_3} > x^{S_3}$, so $z \prec x$. If $x^1 < \eta^1$ then, since $x \notin Q_1(\eta)$, either $x^2 < \eta^2$ or $x^3 < \eta^3$ and therefore $x \in \text{dom}_{S_3} \eta \cup \text{dom}_{S_2} \eta$. We now prove (3) $\text{dom } U \cap [\xi\eta] = \emptyset$ and $\text{dom} [\xi\eta] \cap U = \emptyset$. Let $x \in [\xi\eta]$ and $y \in U$. $y^1 \leq x^1$, $y^2 \geq x^2$ and $y^3 \geq x^3$ therefore $x \prec y$ is impossible and if $y \prec x$ then $y \prec_{S_1} x$, but, since $x \notin \dot{A}^1$, this is also impossible. Combining (1), (2) and (3) it follows that $V = U \cup [\xi\eta]$ solves $Q_1(\xi)$.

LEMMA II.4. *If $\xi \in D^k - \dot{A}^k$ then $Q_k(\xi)$ is solvable and if V solves it then $\text{dom } V \supset T_k(\xi) - V$.*

Proof. W.L.G. $k = 1$. Denote $J = Q_1(\xi) \cap D^1$. Let η be a point where x^1 receives its minimum in J . We show that $\dot{Q}_1^i(\eta) \cap \dot{Q}_1^j(\eta) = \emptyset$ for all $i \neq j$. First, since $\xi \notin \dot{A}^1$, $\dot{A}^1 \cap Q_1(\xi) = \emptyset$. So we have $\dot{Q}_1^1(\eta) \subset \dot{A}^1 \cap Q_1(\eta) \subset \dot{A}^1 \cap Q_1(\xi) = \emptyset$. Next, since x^1 receives its minimum in J at η , $Q_1(\eta) \cap D^1 = \{\eta\}$. $\eta \notin \dot{A}^1$ therefore $\eta \in Q_1(\eta)$ -core. We have $\dot{Q}_1^2(\eta) \cap \dot{Q}_1^3(\eta) \subset Q_1^2(\eta) \cap Q_1^3(\eta) = D^1 \cap Q_1(\eta) = \{\eta\}$, so $\dot{Q}_1^2(\eta) \cap \dot{Q}_1^3(\eta) = \emptyset$. By II.1 $Q_1(\eta)$ is solvable. Let U be a solution of $Q_1(\eta)$; by II.3 $U \cup [\xi\eta]$ solves $Q_1(\xi)$. Now let V be a solution of $Q_1(\xi)$. $\xi \in Q_1(\xi)$ -core so $\xi \in V$. $\text{dom } \xi \supset T_1(\xi) - Q_1(\xi)$ therefore $\text{dom } V \supset (Q_1(\xi) - V) \cup (T_1(\xi) - Q_1(\xi)) = T_1(\xi) - V$.

LEMMA II.5. *Let $S_k = \{i, j\}$. If F^k has c_j -shape and $\mu \in F^k \cap A^k$ then there is a solution V of $Q_i(\mu)$ such that $V \cap F^k = \{\mu\}$.*

Proof. W.L.G. $i = 2$ and $j = 3$. We denote the ends of F^1 by α and β such that $\alpha^2 > \beta^2$. Since F^1 has c_3 -shape we have (1) $\dot{A}^2 \cap R_1(\alpha) = \emptyset$ and (2) $[\beta\alpha] \subset \dot{A}^3$. From (1) it follows that (3) $\dot{Q}_2^2(\mu) = \emptyset$. We show (4) $[\beta\mu] \subset \dot{Q}_2^3(\mu)$. Let $x \in [\beta\mu]$.

$x^1 = \mu^1$, $x^2 < \mu^2$ and $x^3 > \mu^3$. By (2) $x \in \dot{A}^3$, therefore there is a $y \in A^3$ such that $y \succ_{S_3} x$. For small $t > 0$ $z = ty + (1-t)x$ satisfy $z^1 > x^1$, $z^2 > x^2$ and $z^3 > \mu^3$, so $z \in Q_2^3(\mu)$ and $x \in \dot{Q}_2^3(\mu)$. For the relative position of $Q_2^1(\mu)$ and $Q_2^3(\mu)$ we have the following possibilities: (a) $\dot{Q}_2^1(\mu) \cap \dot{Q}_2^3(\mu) = \emptyset$ or (b) $\dot{Q}_2^1(\mu) \cap \dot{Q}_2^3(\mu) \neq \emptyset$. If (a) holds then by (3) $\dot{Q}_2^i(\mu) \cap \dot{Q}_2^j(\mu) = \emptyset$ for all $i \neq j$. By II.1 $Q_2(\mu) - \text{dom } Q_2(\mu)$ is a solution of $Q_2(\mu)$ and since $\mu \in Q_2(\mu)\text{-core}$, $(Q_2(\mu) - \text{dom } Q_2(\mu)) \cap F^1 = \{\mu\}$. If (b) then by I.2 there is an interior point ζ of $\dot{Q}_2^1(\mu) \cap \dot{Q}_2^3(\mu)$. $\zeta^1 > \mu^1$, $\zeta^2 < \mu^2$ and $\zeta^3 > \mu^3$. $\zeta \in D^2 - A^2$, therefore by II.4 $Q_2(\zeta)$ is solvable. If U is a solution of $Q_2(\zeta)$ then, by II.3, $[\zeta\mu] \cup U$ solves $Q_2(\mu)$. Since $([\zeta\mu] \cup U) \cap F^1 = \{\mu\}$ this completes the proof.

DEFINITION II.6. The pair $\{A^i, A^j\}$ satisfies condition M if:

- (1) $\{A^i, A^j\}$ intersects maximally,
- (2) $F^k \cap a^i \cap a^j \not\subset \dot{A}^k$ where $\{k\} = N - \{i, j\}$.

We now formulate the induction hypothesis:

II.7. every 3-P.C.G. G for which $m(G) \leq l - 1$ is solvable. Let G be 3-P.C.G. for which $m(G) = l$. We have to prove that G is solvable. We distinguish between the following possibilities:

II.8. there is at least one pair that satisfies condition M .

II.9. there is no pair that satisfies condition M .

III. Second part: case II.8. W.L.G. $\{A^2, A^3\}$ satisfies condition M . The ends of F^1 will be denoted by α and β such that $\alpha^2 \geq \beta^2$. $F^1 \cap a^2 \cap a^3 \not\subset \dot{A}^1$ therefore at least one of the ends is in $a^2 \cap a^3 - \dot{A}^1$. We shall prove that G is solvable when: (III.1) $\alpha \in a^2 \cap a^3 - \dot{A}^1$. The proof when $\beta \in a^2 \cap a^3 - \dot{A}^1$ is similar to that in case (III.1). We shall distinguish three cases according to the three possible shapes of F^1 in case (III.1).

III.a. F^1 has a -shape. By (III.1) and II.4 $Q_1(\alpha)$ is solvable and if V solves it then (1) $\text{dom } V \supset T_1(\alpha) - V$. Since $\alpha \in F^1$ $\dot{A}^2 \cap \dot{A}^3 \cap R_1(\alpha) = \emptyset$. By I.19 $\dot{R}_1^2(\alpha) \cap \dot{R}_1^3(\alpha) = \emptyset$, so $\{R_1^2(\alpha), R_1^3(\alpha)\}$ does not intersect maximally. From I.20 it follows now that $m(R_1(\alpha)) \leq l - 1$. By II.7 $R_1(\alpha)$ is solvable. If $Q_1(\alpha)$ and $R_1(\alpha)$ are independent then from I.22 and (1) it follows that A has a solution. If $Q_1(\alpha)$ and $R_1(\alpha)$ are not independent then if V solves $Q_1(\alpha)$ and W solves $R_1(\alpha)$ either (2) $\text{dom } V \cap W \neq \emptyset$ or (3) $\text{dom } W \cap V \neq \emptyset$. From III.a and I.17 it follows that (4) $\text{dom } R_1(\alpha) \cap Q_1(\alpha) = \emptyset$. By (4) we have that (3) is impossible. By (III.1) $\alpha \in A - \text{dom } A$ therefore (5) $\alpha \in W \cap V$. From (2), (5) and I.12 it follows that there is a $z \neq \alpha$ in $V \cap A^1$. By I.15 and due to III.a, we may assume that every $y \in A^1$ satisfies $y^2 \leq \alpha^2$. Let ζ be a point where x^3 receives its maximum⁽²⁾ in $V \cap A^1$. If $u \in V \cap A^1$ then $\zeta^2 = u^2$ and $\zeta^3 \geq u^3$ and therefore (6) $\text{dom}_{S_1} \zeta \supset \text{dom}_{S_1} u$. $\alpha \in D^3 - \dot{A}^3$ therefore, by II.4, $Q_3(\alpha)$ is solvable. If U solves $Q_3(\alpha)$ then by (4) we have that (7) $\text{dom } U \cap V = \emptyset$. We remark that (8) $Q_3(\alpha) \cap A^3 \subset \{x: x^2 = \alpha^2\}$. Let x^1 receive its maximum in $U \cap A^3$ at the point η . We define: $v = (\eta^1, e - \eta^1 - \zeta^3, \zeta^3)$. By I.12 and (6) we have that (9) $R_1(\alpha) - \text{dom } V = R_1(\alpha)$

(2) Observe that a solution of a compact set is compact. see [3, Theorem 3].

— $\text{dom } s_1 \zeta = Q_3(\alpha) \cup \{x: x \in R_1(\alpha), x^3 \geq \zeta^3\}$. By (8) we have that (10) $Q_2(\alpha) - \text{dom } U = Q_2(\alpha) - \text{dom } s_3 \eta = \{x: x \in Q_2(\alpha), x^1 \geq \eta^1\}$. Combining (1), (9) and (10) we have (11) $A - \text{dom}(U \cup V) = \{x: x \in A, x^{s_2} \geq v^{s_2}\}$. If $v \notin A$ then from I.4, (11), (9) and (7) it follows that $U \cup V$ solves A . Suppose now that $v \in A$. We define $v_1 = (\alpha^1, e - \alpha^1 - \zeta^3, \zeta^3)$. By I.16 we have that (12) $\emptyset = A^2 \cap Q_2(v_1) \supset A^2 \cap Q_2(v)$. $Q_1(\alpha) \cup Q_3(\alpha) \subset R_2(v)$ therefore by (12) and I.12 we have (13) $\text{dom } Q_2(v) \cap (Q_1(\alpha) \cup Q_3(\alpha)) = \emptyset$. $v \in D^2 - A^2$ therefore $Q_2(v)$ is solvable. If U_1 solves $Q_2(v)$ then by (11) and (13) $V \cup U \cup U_1$ is a solution of A .

III.b. F^1 has b -shape. Due to III.b, we have that $R_1(\alpha) \cap (\dot{A}^2 \cup \dot{A}^3) = \emptyset$ and therefore (1) $\dot{R}_1^2(\alpha) = \dot{R}_1^3(\alpha) = \emptyset$. From (1) it follows that (2) $R_1^i(\alpha) \cap \dot{R}_1^j(\alpha) = \emptyset$ for all $i \neq j$. If $x \in R_1(\alpha)$ then $x^2 + x^3 \leq \alpha^2 + \alpha^3$ therefore (3) $[\alpha\beta] \cap \text{dom } s_1 R_1(\alpha) = \emptyset$. From (1) and (3) we have that (4) $R_1(\alpha) - \text{dom } R_1(\alpha) \supset [\alpha\beta]$. We also have that (5) $T_1(\alpha) - \text{dom } R_1(\alpha) = T_1(\alpha) - \text{dom } \{\alpha, \beta\} = \{x: x \in T_1(\alpha), x^2 \geq \alpha^2, x^3 \geq \beta^3\}$. Define $\mu = (e - \alpha^2 - \beta^3, \alpha^2, \beta^3)$.

III.b.1. $\mu \notin A$. By I.4 we have that (6) $\{x: x^{s_1} \geq \mu^{s_1}\} \cap A = \emptyset$. By (2) and II.1 the $R_1(\alpha)$ -core is the solution of $R_1(\alpha)$. By (4), (5) and (6) we have that $R_1(\alpha) - \text{dom } R_1(\alpha)$ solves A .

III.b.2. $\mu \in A$. $\mu \in D^1 - \dot{A}^1$ therefore $Q_1(\mu)$ is solvable. We remark that (7) $Q_1(\mu) \cap A^1 \subset \{x: x^2 = \alpha^2\}$. We distinguish several subcases of II.b.2.

III.b.2.1. There is a solution V_1 of $Q_1(\mu)$ such that $V_1 \cap A^1 = \emptyset$. We have that (8) $\text{dom } V_1 \cap R_1(\alpha) = \emptyset$. From (5) and (8) it follows that

$$V_1 \cup (R_1(\alpha) - \text{dom } R_1(\alpha))$$

is a solution of G .

III.b.2.2. There is a solution V_2 of $Q_1(\mu)$ such that $V_2 \cap A^1 = \{\mu\}$. In this case: $R_1(\alpha) - \text{dom } V_2 = Q_3(\alpha) \cup Q_2(\beta)$. $\alpha \in D^3 - \dot{A}^3$ and $\beta \in D^2 - \dot{A}^2$ so $Q_3(\alpha)$ and $Q_2(\beta)$ are solvable and if U solves $Q_3(\alpha)$ and W solves $Q_2(\beta)$ then, by I.13, $\alpha \in U$ and $\beta \in W$. Since $Q_3(\alpha) \subset R_2(\beta)$ and $Q_2(\beta) \subset R_3(\alpha)$ it follows from I.12 that

$$\text{dom } U \cap W = \text{dom } W \cap U = \emptyset.$$

From these results and (5) it follows that $U \cup W \cup V_2$ is a solution of G .

III.b.2.3. There is a solution V_3 of $Q_1(\mu)$ such that $V_3 \cap A^1 - \{\mu\} \neq \emptyset$. Let x^3 receive its maximum in $V_3 \cap A^1$ at the point ζ . $\zeta^2 = \mu^2$ and $\zeta^3 > \mu^3$. By II.3 $[\mu\alpha] \cup V_3 = V'_3$ solves $Q_1(\alpha)$. We define $v = (\alpha^1, e - \alpha^1 - \zeta^3, \zeta^3)$. $R_1(\alpha) - \text{dom } V'_3 = Q_3(\alpha) \cup \{x: x \in A, x^{s_2} \geq v^{s_2}\}$. Let U solve $Q_3(\alpha)$. $\text{dom } U \cap (Q_1(\alpha) \cup \{x: x \in A, x^{s_2} \geq v^{s_2}\}) = \emptyset$. If $v \notin A$ then $V'_3 \cup U$ solves G . If $v \in A$ then $v \in D^2 - A^2$. By I.16 $Q_2(v) \cap A^2 = \emptyset$. $Q_1(\alpha) \cup Q_3(\alpha) \subset R_2(v)$ therefore by I.12 $\text{dom } Q_2(v) \cap (Q_1(\alpha) \cup Q_3(\alpha)) = \emptyset$. If W solves $Q_2(v)$ we have that $V'_3 \cup U \cup W$ is a solution of G .

III.c. F^1 has c_3 -shape. As in III.a, we have that $R_1(\alpha)$ is solvable and if $Q_1(\alpha)$ and $R_1(\alpha)$ are independent then G is solvable. If $Q_1(\alpha)$ and $R_1(\alpha)$ are not independent and V solves $Q_1(\alpha)$ and W solves $R_1(\alpha)$ then either $\text{dom } V \cap W \neq \emptyset$ or $\text{dom } W \cap V \neq \emptyset$.

III.c.1. There exist V_0 and W_0 such that $\text{dom } V_0 \cap W_0 \neq \emptyset$. $\alpha \in W_0$ therefore there must be a $z \neq \alpha$ in $V_0 \cap A^1$. We have that either $z^3 = \alpha^3$ or $z^2 = \alpha^2$.

III.c.1.1. $z^3 = \alpha^3$. In this case we have that (1) $y \in A^1$ implies $y^3 \leq z^3$. Let x^2 take its maximum in $V_0 \cap A^1$ at the point ζ . Define $v = (\alpha^1, \zeta^2, e - \alpha^1 - \zeta^2)$. From (1) it follows that (2) $R_1(\alpha) - \text{dom } V_0 = Q_2(\alpha) \cup \{x: x \in A, x^{S_3} \geq v^{S_3}\}$. By II.5 there is a solution U of $Q_2(\alpha)$ such that $U \cap F^1 = \{\alpha\}$; it follows that (3) $\text{dom } U \cap Q_1(\alpha) = \emptyset$. If $v \notin A$ then $U \cup V_0$ is a solution of G . If $v \in A$ then $v \in D^3 - A^3$, so $Q_3(v) \cap A^3 = \emptyset$. It follows that (4) $\text{dom } Q_3(v) \cap (Q_2(\alpha) \cup Q_1(\alpha)) = \emptyset$. We also have that (5) $\text{dom } U \cap Q_3(v) = \emptyset$. Now if U_1 is a solution of $Q_3(v)$ then, combining (2), (3), (4) and (5), we have that $V_0 \cup U \cup U_1$ is a solution of G .

III.c.1.2. $z^2 = \alpha^2$. We now have that $y \in A^1$ implies $y^2 \leq \alpha^2$. We show that we may suppose: (*) there is no $u \in A^3$ such that $u^2 = \alpha^2$ and $u^1 > \alpha^1$. If (*) fails then F^2 has b -shape and $\{A^1, A^3\}$ satisfies condition M , so by III.b G is solvable. We also notice that: (**) if $\vartheta \in A$, $\vartheta^2 = \alpha^2$, $\vartheta^3 \leq \beta^3$ and J solves $Q_1(\vartheta)$ then $J \cup [\alpha\vartheta]$ solves $Q_1(\alpha)$. Now if (*) holds and U is a solution of $Q_3(\alpha)$ then $\text{dom } U \cap (Q_1(\alpha) \cup Q_2(\alpha)) = \emptyset$. Let V solve $Q_1(\alpha)$. We denote by $\mu(V)$ the point where x^3 takes its maximum in $A^1 \cap V$. The point where x^3 takes its maximum in $A^1 \cap \{x: x^1 = \alpha^1\}$ is denoted by ζ . U denotes a fixed solution of $Q_3(\alpha)$. We remark that $\alpha \in U$.

III.c.1.2.1. There is a solution V_1 of $Q_1(\alpha)$ such that $\beta^3 < \mu^3(V_1)$. Define $v = (\alpha^1, e - \alpha^1 - \mu^3(V_1), \mu^3(V_1))$. We have that $R_1(\alpha) - \text{dom } V_1 = Q_3(\alpha) \cup \{x: x \in A, x^{S_2} \geq v^{S_2}\}$. If $v \notin A$ then $U \cup V_1$ is a solution of G . If $v \in A$ then $v \in D^2 - A^2$. Let U_1 solve $Q_2(v)$. $\text{dom } U_1 \cap (Q_1(\alpha) \cup Q_3(\alpha)) = \emptyset$, so $U \cup U_1 \cup V_1$ is a solution of G .

III.c.1.2.2. Every solution V of $Q_1(\alpha)$ satisfies $\mu^3(V) \leq \beta^3$ and there is a solution V_1 of $Q_1(\alpha)$ such that $\mu^3(V_1) = \beta^3$. $\beta \in a^2 \cap D^2$ therefore $Q_2(\beta)$ is solvable. If U_1 solves $Q_2(\beta)$ then $U_1 \cap A^2 = \{\beta\}$ and $\text{dom } U_1 \cap (Q_1(\mu(V_1)) \cup Q_3(\alpha)) = \emptyset$. Let U_2 be a solution of $Q_1(\mu(V_1))$. From (**) it follows that $U_2 \cap A^1 = \{\mu(V_1)\}$. So we have that $U \cup U_1 \cup U_2$ is a solution of G .

III.c.1.2.3. If V solves $Q_1(\alpha)$ then $\mu^3(V) < \beta^3$.

III.c.1.2.3.1. $\zeta^3 \geq \beta^3$. Define $v = (e - \alpha^2 - \beta^3, \alpha^2, \beta^3)$. By I.17 $Q_1(\alpha) \cap \text{dom } R_1(\alpha) = \text{dom } [\alpha\beta] \cap Q_1(\alpha)$, so $\text{dom } R_1(\alpha) \cap \{x: x \in A, x^{S_1} \geq v^{S_1}\} = \emptyset$. If $v \notin A$ and U_1 solves $Q_2(\beta)$ then $U \cup [\alpha\beta] \cup U_1$ solves G . If $v \in A$ let U_2 be a solution of $Q_1(v)$. By (**) and the definition of U_2 $U_2 \cap A^1 = \emptyset$. So in this case $U \cup U_1 \cup U_2 \cup [\alpha\beta]$ solves G .

III.c.1.2.3.2. $\zeta^3 < \beta^3$. Define $v = (e - \alpha^2 - \zeta^3, \alpha^2, \zeta^3)$. By II.5 there is a solution U_1 of $Q_2(\zeta)$ such that $U_1 \cap F^1 = \{\zeta\}$. If $v \notin A$ then $U \cup U_1 \cup [\alpha\zeta]$ is a solution of G . If $v \in A - A^1$ and U_2 solves $Q_1(v)$ then $U \cup U_1 \cup U_2 \cup [\alpha\zeta]$ is a solution of G . If $v \in A^1$ then $U_1 \cup U_2 \cup U$ is a solution of G .

III.c.2. If V solves $Q_1(\alpha)$ and W $R_1(\alpha)$ then $\text{dom } W \cap V \neq \emptyset$ and $\text{dom } V \cap W = \emptyset$. By I.17 we have that $\text{dom } (R_1(\alpha) - [\beta\alpha]) \cap Q_1(\alpha) = \emptyset$. By I.19 $\hat{R}_1^3(\alpha) = \hat{A}^3 \cap R_1(\alpha)$. $[\beta\alpha] \subset \hat{A}^3$ therefore $[\beta\alpha] \subset \hat{R}_1^3(\alpha)$. So we conclude that

$$(1) \quad \text{dom } (R_1(\alpha) - \text{dom } R_1(\alpha)) \cap Q_1(\alpha) = \emptyset.$$

Since F^1 has c_3 -shape we have also that $\dot{A}^2 \cap R_1(\alpha) = \emptyset$. If $\{A^1, A^3\}$ does not intersect maximally then, by I.20, $\{R_1^1(\alpha), R_1^3(\alpha)\}$ does not intersect maximally and we have that $m(R_1(\alpha)) = 0$. By II.2 the $R_1(\alpha)$ -core is the solution of $R_1(\alpha)$ which, by (1), contradicts III.c.2. So $\{A^1, A^3\}$ intersects maximally. The x^2 coordinate of F^2 will be denoted by d .

III.c.2.1. $d < \alpha^2$. We denote the ends of F^2 by γ and δ such that $\delta^1 \geq \gamma^1$. We remark that $\delta \in D^2 - A^2$. So $Q_2(\delta)$ is solvable and if U solves it then $\text{dom } U \supset T_2(\delta) - U$ and $\text{dom } U \cap R_2(\delta) = \emptyset$. Denote $v = (\alpha^1, d, e - d - \alpha^1)$ and $P = \{x: x \in A, x^{S_3} \geq v^{S_3}\}$. We remark that $\dot{P}^i \cap \dot{P}^j = \emptyset$ for all $i \neq j$ and that $(P - \text{dom } P) \cap ((\alpha\beta] \cup [\gamma\delta)) = \emptyset$. So⁽³⁾ $P - \text{dom } P$ solves P and

$$\text{dom}(P - \text{dom } P) \cap (Q_2(\delta) \cup Q_1(\alpha)) = \emptyset.$$

Summing we have that $U_1 = U \cup (P - \text{dom } P)$ solves $R_1(\alpha)$ and that $\text{dom } U_1 \cap Q_1(\alpha) = \emptyset$. Since this result contradicts III.c.2 $d < \alpha^2$ is impossible.

III.c.2.2. $d = \alpha^2$. If F^2 has a -shape or b -shape then $\{A^1, A^3\}$ satisfies condition M and by III.a or III.b G is solvable. It remains only to complete the proof when F^2 has c_3 or c_1 -shape.

III.c.2.2.1. $\gamma^1 < \alpha^1$. In this case F^2 has c_3 -shape. Let η satisfy $\eta^2 = \alpha^2$ and $\alpha^1 > \eta^1 > \max.(\gamma^1, e - \alpha^2 - \beta^3)$ and U_1 be a solution of $Q_1(\eta)$. $U_2 = [\alpha\eta] \cup U_1$ solves $Q_1(\alpha)$. Let $\zeta \in (\alpha\beta] \cap A^1$, U_3 be a solution of $Q_2(\zeta)$ and U_4 a solution of $Q_3(\alpha)$. $U_5 = U_3 \cup U_4 \cup [\alpha\zeta]$ is a solution of $R_1(\alpha)$. But $\text{dom } U_2 \cap U_5 \neq \emptyset$ contradicting III.c.2, so III.c.2.2.1 is impossible.

III.c.2.2.2. $\gamma^1 = \alpha^1$. In this case F^2 has c_1 -shape. By II.5 there exist solutions U_1 of $Q_3(\alpha)$ and U_2 of $Q_2(\alpha)$ such that $U_1 \cap F^2 = \{\alpha\} = U_2 \cap F^1$. $U = U_1 \cup U_2$ is a solution of $R_1(\alpha)$ but⁽³⁾ $\text{dom } U \cap Q_1(\alpha) = \emptyset$ contradicting III.c.2.

III.c.2.2.3. $\gamma^1 > \alpha^1$. In this case F^2 has c_1 -shape and $\gamma \in a^1 \cap a^3 - A^2$. Let U solve $Q_2(\gamma)$. $U_1 = U \cup (Q_3(\alpha) - \text{dom } Q_3(\alpha))$ solves $R_1(\alpha)$ but $\text{dom } U_1 \cap Q_1(\alpha) = \emptyset$ which is impossible.

III.c.2.3. $d > \alpha^2$. In this case we show that $\dot{A}^1 \cap \dot{A}^3 \cap R_1(\alpha) = \emptyset$. It follows that $\{R_1^1(\alpha), R_1^3(\alpha)\}$ does not intersect maximally which is impossible as we have already seen. Suppose that $\dot{A}^1 \cap \dot{A}^3 \cap R_1(\alpha) \neq \emptyset$. Let $x \in \dot{A}^1 \cap \dot{A}^3 \cap R_1(\alpha)$. $x^1 \geq \alpha^1$. Since⁽⁴⁾ $\alpha \in a^3$ $x^2 < \alpha^2$. Let $z \in a^1 \cap a^3 \cap F^2$ and y be an interior point of $\dot{A}^1 \cap \dot{A}^3$. There is a $u \in [yz] \cap \dot{A}^1 \cap \dot{A}^3$ such that $u^2 > \alpha^2$. So there is a $w \in [ux]$ for which $w^2 = \alpha^2$. $w \in \dot{A}^1 \cap \dot{A}^3$. If $w^1 \geq \alpha^1$ then $\alpha \in \dot{A}^3$ and if $w^3 \geq \alpha^3$ then $\alpha \in \dot{A}^1$. Since both cases are impossible we must have $\emptyset = \dot{A}^1 \cap \dot{A}^3 \cap R_1(\alpha)$.

IV. Third part: case II.9.

IV.1. $m(G) \leq 2$. W.L.G. $\{A^2, A^3\}$ does not intersect maximally.

IV.1.1. $\dot{A}^2 \cap \dot{A}^3 = \emptyset$. We shall show that $m(G) = 0$. If $m(G) > 0$ then, W.L.G., $\{A^1, A^3\}$ intersects maximally. Let $z \in a^1 \cap a^3 \cap F^2$ and y be an interior point of $\dot{A}^1 \cap \dot{A}^3$. $[yz] \subset \dot{A}^3$. If $z \in \dot{A}^2$ then we have $\dot{A}^2 \cap \dot{A}^3 \neq \emptyset$ which is impossible. If

(3) See I.17.

(4) If $x^2 \geq \alpha^2$ then $\alpha^{S_3} \geq x^{S_3}$ and $x \in \dot{A}^3$ imply $\alpha \in \dot{A}^3$ which is untrue.

$z \notin \dot{A}^2$ then $\{A^1, A^3\}$ satisfies condition M , which is again impossible. Therefore $\{A^1, A^3\}$ does not intersect maximally.

IV.1.2. $\dot{A}^3 \supset A^2$. If $\{A^1, A^3\}$ intersects maximally then there is a $z \in a^1 \cap a^3 \cap F^2$. $a^3 \cap \dot{A}^2 = \emptyset$; therefore $z \notin \dot{A}^2$. But since we have $a^1 \cap a^3 \cap F^2 \subset \dot{A}^2$, $\{A^1, A^3\}$ cannot intersect maximally.

IV.1.2.1. $\dot{A}^1 \cap \dot{A}^3 = \emptyset$. $\dot{A}^2 \cap \dot{A}^1 \subset \dot{A}^1 \cap A^2 \subset \dot{A}^1 \cap \dot{A}^3 = \emptyset$, so $m(G) = 0$.

IV.1.2.2. $\dot{A}^3 \supset A^1$. We denote $C = A - \text{dom } A$. If $x \in A - C$ then there is a $y \in A$ such that $y \succ_{s_3} x$. So there is a $z \in a^3$ such that $z \succ x$. $a^3 \cap (\dot{A}^1 \cup \dot{A}^2) = \emptyset$ therefore $z \in C$. By 2.6 C is a solution of G .

IV.1.2.3. $\dot{A}^1 \supset A^3$. We have $\dot{A}^1 \supset A^3 \supset \dot{A}^3 \supset A^2$, so $m(G) = 0$.

IV.1.3. $\dot{A}^2 \supset A^3$. The proof in this case parallels that in IV.1.2.

IV.2. $m(G) = 3$. We denote $F^k = [\alpha_k \beta_k]$ and $D = \bigcap_{h=1}^3 A^h$.

LEMMA IV.2.1. *Under the assumptions of IV.2 we can find i and k such that $S_k = \{i, j\}$ and:*

(1) $\alpha_k^i \geq \beta_k^i$,

(2) $\alpha_k \in a^i \cap a^j$,

(3.a) x^j takes its maximum in $\{x: x^i = \alpha_k^i\} \cap D$ at a point $\vartheta \in a^k$ such that every $y \in A^k$ that satisfies $y^j = \vartheta^j$ and $y^i > \vartheta^i$ is in $\dot{A}^i$, or

(3.b) x^i takes its maximum in $\{x: x^j = \alpha_k^j\} \cap D$ at a point $\rho \in a^k \cap \dot{A}^j$ and F^k has c_j -shape.

Proof. $F^1 = [\alpha_1 \beta_1]$. W.L.G. $\alpha_1^2 \geq \beta_1^2$. We also suppose that $\alpha \in a^2 \cap a^3$. If $\alpha \notin a^2 \cap a^3$ then $\beta \in a^2 \cap a^3$ and the proof is not altered much. We now consider F^3 .

IV.2.1.1. Every $y \in F^3$ satisfies $y^2 < \alpha_1^2$. Let ϑ be the point where x^3 takes its maximum in $\{x: x^2 = \alpha_1^2\} \cap D$. $\vartheta \neq \alpha_1$. We show that $\vartheta \in \dot{A}^2$. If $F^3 = [\alpha_3 \beta_3]$ and $\alpha_3^1 \geq \beta_3^1$ then $\alpha_3^3 > \vartheta^3$ since $\vartheta \notin F^3$. If $\alpha_3^1 > \vartheta^1$ then $\alpha_3 \succ_{s_2} \vartheta$, and if $\vartheta^1 \geq \alpha_3^1$ then there is a $u \in [\alpha_1 \alpha_3]$ such that $u \succ_{s_2} \vartheta$. Now if $\vartheta \in a^3$ then F^2 has c_1 -shape and $\vartheta \in a^1$ and if $\vartheta \in \dot{A}^3$ then it follows from I.3 that $\vartheta \in a^1$. So in this case we can choose $k = 1$ and $i = 2$.

IV.2.1.2. There is a $y \in F^3$ such that $y^2 = \alpha_1^2$. Let $\vartheta = y$. If F^3 has a -shape then $\vartheta \in a^1$ and there is no $u \in A^1$ that satisfies $u^3 = \vartheta^3$ and $u^2 > \vartheta^2$. So we can choose $k = 1$ and $i = 2$. If F^3 has not a -shape then $\beta_3^2 > \alpha_3^2$. If F^3 has b -shape and $y = \beta_3$ then there is no $u \in A^1$ such that $u^3 = \vartheta^3$, and $u^2 > \vartheta^2$ and we can choose $k = 1$ and $i = 2$. If $y \neq \beta_3$ then we have that $\beta_3 \in a^2 \cap a^1$ and every $x \in F^1$ satisfies $x^2 < \beta_3^2$. By IV.2.1.1. we may take $k = 3$ and $i = 2$. If F^3 has c_2 -shape then $\vartheta \in a^1$ and if $u \in A$, $u^3 = \vartheta^3$ and $u^2 > \vartheta^2$ then $u \in \dot{A}^2$. So we can take $k = 1$ and $i = 2$. If F^3 has c_1 -shape and $y = \beta_3$ we choose $k = 1$ and $i = 2$. If $y \neq \beta_3$ then we have that $\beta_3 \in a^2 \cap a^1$ and every $x \in F^1$ satisfies $x^2 < \beta_3^2$. By IV.2.1.1. we can choose $k = 3$ and $i = 2$.

IV.2.1.3. Every $y \in F^3$ satisfies $y^2 > \alpha_1^2$. If F^3 has a , b or c_1 -shape then we have that $\beta_3^2 \geq \alpha_3^2$, $\beta_3 \in a^2 \cap a^1$ and every $x \in F^1$ satisfies $x^2 < \beta_3^2$. By IV.2.1.1, we can choose $k = 3$ and $i = 2$. Now suppose that F^3 has c_2 -shape. Let x^1 take its maxi-

mum in $D \cap \{x: x^2 = \alpha_3^2\}$ at the point ρ . $\rho \neq \alpha_3$. We shall show that $\rho \in \dot{A}^1 \cap \dot{A}^2$. $\rho^2 = \alpha_3^2 < \beta_3^2$. $\rho^1 > \alpha_3^1$ therefore $\rho^3 < \alpha_3^3 = \beta_3^3$. So $\beta_3 \succ_{s_1} \rho$. We have also that $\alpha_1^1 > \rho^1$ therefore there is a $u \in [\alpha_1 \alpha_3]$ such that $u \succ_{s_2} \rho$. It follows from I.3 that $\rho \in a^3$. Summing we have: $\alpha_3^1 \geq \beta_3^1$, $\alpha_3 \in a^2 \cap a^1$ and x^1 takes its maximum in $D \cap \{x: x^2 = \alpha_3^2\}$ at a point $\rho \in a^3 \cap \dot{A}^2$. So we can take $k = 3$ and $i = 1$.

We now prove that G is solvable in case IV.2. W.L.G. the results of IV.2.1 hold for $k = 1$ and $i = 2$.

IV.2.2. (3.a) holds in IV.2.1. We remark⁽⁵⁾ that if $z \in A^1 \cap Q_1(\vartheta)$ then $z^3 = \vartheta^3$. If there is a $z \in A^1$ such that $z^3 = \vartheta^3$ and $z^2 > \vartheta^2$ then F^3 has c_2 -shape. By Lemma II.5 there is a solution V of $Q_1(\vartheta)$ such that $V \cap F^3 = \{\vartheta\}$. So we can always find a solution V_1 of $Q_1(\vartheta)$ such that $V_1 \cap A^1 = \{\vartheta\}$. Similar reasoning shows that there is always a solution V_2 of $Q_3(\alpha_1)$ such that $V_2 \cap A^3 = \{\alpha_1\}$.

IV.2.2.1. $\vartheta^3 > \beta_1^3$. We define $v = (\alpha_1^1, e - \alpha_1^1 - \vartheta^3, \vartheta^3)$. Suppose $v \in A$. We have that $Q_2(v) \cap A^2 = \emptyset$. So if U is a solution of $Q_2(v)$ then $V_1 \cup V_2 \cup U \cup [\vartheta \alpha_1]$ solves G . If $v \notin A$ then $V_1 \cup V_2 \cup [\vartheta \alpha_1]$ solves G .

IV.2.2.2. $\vartheta^3 = \beta_1^3$. If U is a solution of $Q_2(\beta_1)$ then $V_1 \cup V_2 \cup U$ is a solution of G .

IV.2.2.3. $\vartheta^3 < \beta_1^3$. Let x^3 take its maximum in $A^1 \cap \{x: x^1 = \alpha_1^1\}$ at ζ . We define $\mu = (e - \alpha_1^1 - \beta_1^3, \alpha_1^1, \beta_1^3)$. Suppose $\zeta^3 \geq \beta_1^3$ and $\mu \in A$. In this case if W is a solution of $Q_2(\beta_1)$ and W_1 is a solution of $Q_1(\mu)$ then $V_2 \cup W \cup W_1 \cup F^1$ solves G . If $\mu \notin A$ then $V_2 \cup W \cup F^1$ is a solution of G . If $\zeta^3 < \beta_1^3$ then F^1 has c_3 -shape. We define $\eta = (e - \zeta^3 - \alpha_1^2, \alpha_1^2, \zeta^3)$. By II.5 there is a solution U of $Q_2(\zeta)$ such that $\{\zeta\} = U \cap F^1$. If $\eta \notin A$ then $U \cup V_2 \cup [\zeta \alpha_1]$ solves G . If $\eta \in A - A^1$ and U_1 solves $Q_1(\eta)$ then $V_2 \cup U \cup U_1 \cup [\zeta \alpha_1]$ is a solution for G . If $\eta \in A^1$ then $\eta = \vartheta$ and $V_1 \cup V_2 \cup U$ is a solution of G .

IV.2.3. (3.b) holds in IV.2.1. So F^1 has c_3 -shape. By II.5 there is a solution V of $Q_2(\alpha_1)$ such that $V \cap F^1 = \{\alpha_1\}$. Next we show that there is a solution V_1 of $Q_1(\rho)$ such that $V_1 \cap A^1 = \{\rho\}$. If $Q_1(\rho) \cap A^1 = \{\rho\}$ this follows from the fact that ρ belongs to every solution of $Q_1(\rho)$. If there is $x \in Q_1(\rho) \cap A^1$, $x \neq \rho$, then $x \in a^1$ and⁽⁶⁾ $x^2 = \rho^2$. Using I.10 and observing that $\rho \in \dot{A}^3$ we see that F^2 has C_3 -shape and $\rho \in F^2$. II.5 yields a desired V_1 . Now define $v = (\alpha_1^1, \rho^2, e - \alpha_1^1 - \rho^2)$. Observe that $Q_3(v) \cap A^3 = \emptyset$. If $v \in A$ and V_2 is a solution of $Q_3(v)$ then $V \cup V_1 \cup V_2 \cup [\rho \alpha_1]$ is a solution of G . If $v \notin A$ then $V \cup V_1 \cup [\rho \alpha_1]$ solves G .

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(5) Suppose there is $z \in A^1 \cap Q_1(\vartheta)$ with $z^3 > \vartheta^3$, $\alpha_1 \in \dot{A}^1$ so there is $u \in A^1$ with $u^2 > \alpha_1^2 = \vartheta^2$. For small $t > 0$ $y = tu + (1-t)z$ satisfy $y^{S_1} > \vartheta^{S_1}$ and $y \in A^1$ which is impossible since $\vartheta \in a^1$.

(6) By an argument similar to that in footnote (5).